Positive temperature versions of two theorems on first-passage percolation

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Abstract

The estimates on the fluctuations of first-passsage percolation due to Talagrand and Benjamini–Kalai–Schramm are transcribed into the positive-temperature setting of random Schrödinger operators.

1 Introduction

Let $H = -\frac{1}{2d}\Delta + V$ be a random Schrödinger operator on $\mathbb{Z}^d$ with non-negative potential $V \geq 0$:

$$(H\psi)(x) = (1 + V(x))\psi(x) - \frac{1}{2d}\sum_{y\sim x}\psi(y) \ , \quad \psi \in \ell^2(\mathbb{Z}^d) \ .$$

Assume that the entries of V are independent, identically distributed, and satisfy

$$\mathbb{P}\{V(x) > 0\} > 0 \ . \tag{1}$$

The inverse $G = H^{-1}$ of H defines a random metric

$$\rho(x, y) = \log \frac{\sqrt{G(x,x)G(y,y)}}{G(x,y)} \tag{2}$$

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on $\mathbb{Z}^d$ (see Lemma 2.4 below for the verification of the triangle inequality). We are interested in the behaviour of $\rho(x, y)$ for large $\|x-y\|$ (here and forth $\|\cdot\|$ stands for the ℓ_1 norm); to simplify the notation, set $\rho(x) = \rho(0, x)$.

Zerner proved [16, Theorem A], using Kingman's subadditive ergodic theorem [10], that if V satisfies (1) and

$$\mathbb{E} \log^d(1 + V(x)) < \infty \ . \tag{3}$$

then

$$\rho(x) = \|x\|_V (1 + o(1)) \ , \quad \|x\| \to \infty \ , \tag{4}$$

where $\|\cdot\|_V$ is a deterministic norm on $\mathbb{R}^d$ determined by the distribution of V. As to the fluctuations of $\rho(x)$, Zerner showed [16, Theorem C] that (1), (3), and

$$\text{if } d = 2, \text{ then } \mathbb{P}\{V(x) = 0\} = 0$$

imply the bound

$$\operatorname{Var} \rho(x) \leq C_V \|x\| \ . \tag{5}$$

In dimension $d = 1$, the bound (5) is sharp; moreover, ρ obeys a central limit theorem

$$\frac{\rho(x) - \mathbb{E}\rho(x)}{\sigma_V |x|^{1/2}} \xrightarrow[|x|\to\infty]{D} N(0, 1) \ ,$$

which follows from the results of Furstenberg and Kesten [8]. In higher dimension, the fluctuations of ρ are expected to be smaller: the exponent

$$\chi_d = \limsup_{\|x\|\to\infty} \frac{\frac{1}{2} \log \operatorname{Var} \rho(x)}{\log \|x\|}$$

is expected to be equal to $1/3$ in dimension $d = 2$, and to be even smaller in higher dimension.

These conjectures are closely related to the corresponding conjectures for first-passage percolation. In fact, ρ is a positive-temperature counterpart of the (site) first-passage percolation metric corresponding to $\omega = \log(1 + V)$; we refer to Zerner [16, Section 3] for a more elaborate discussion of this connection.

The rigorous understanding of fluctuations in dimension $d \geq 2$ is for now confined to a handful of integrable models (see Corwin [7] for a review); extending it beyond this class remains a major open problem. We refer to the works of Chatterjee [6] and Auffinger–Damron [1, 2] for some recent results.

Here we carry out a much more modest task: verifying that the bounds on the fluctuations in (bond) first-passage parcolation due to Talagrand [15] and Benjamini–Kalai–Schramm [4] are also valid for the random matric (2). Zerner's bound (5) is a positive-temperature counterpart of Kesten's estimate [9]. Kesten showed that the (bond) first-passage percolation ρ_{FPP} satisfies

$$\operatorname{Var} \rho_{\mathrm{FPP}}(x) \leq C\|x\| \ ;$$

furthermore, if the underlying random variables have exponential tails, then so does $(\rho_{\mathrm{FPP}}(x) - \mathbb{E}\rho_{\mathrm{FPP}}(x))/\sqrt{\|x\|}$. Talagrand improved the tail bound to

$$\mathbb{P}\left\{|\rho_{\mathrm{FPP}}(x) - \mathbb{E}\rho_{\mathrm{FPP}}(x)| \geq t\right\} \leq C \exp\left\{-\frac{t^2}{C\|x\|}\right\} \ , \quad 0 \leq t \leq \|x\| \ .$$

Benjamini, Kalai, and Schramm [4] proved, in dimension $d \geq 2$, the sublinear bound

$$\operatorname{Var} \rho_{\mathrm{FPP}}(x) \leq C\|x\| / \log(\|x\| + 2) \ ,$$

for the special case of Bernoulli-distributed potential. Benaïm and Rossignol [3] extended this bound to a wider class of distributions ("nearly gamma" in the terminology of [3]), and complemented it with an exponential tail estimate. Extensions of the Benjamini–Kalai–Schramm bound to other models have been found by van der Berg and Kiss [5], and by Matic and Nolen [12].

Theorem 1 below is a positive temperature analogue of Talagrand's bound (with a slightly stronger conclusion under a slightly stronger assumption – mainly, to use a more elementary concentration inequality from [13, 15] instead of a more involved one from [15]), and Theorem 2 – of the Benjamini–Kalai–Schramm bound. The strategy of the proof is very close to the original arguments; the modification mainly enters in a couple of deterministic estimates. Set $\mu(x) = \mathbb{E}\rho(x)$.

Theorem 1. *Suppose the entries of V are independent, identically distributed, and bounded from below by $\epsilon > 0$. Also assume that the entries of V are bounded from above by $0 < M < \infty$. Then*

$$\mathbb{P}\left\{\rho(x) \leq \mu(x) - t\right\} \leq C \exp\left\{-\frac{t^2}{C(\epsilon, M)(\mu(x) + 1)}\right\} \ , \tag{6}$$

and

$$\mathbb{P}\left\{\rho(x) \geq \mu(x) + t\right\} \leq C \exp\left\{-\frac{t^2}{C(\epsilon, M)(\mu(x) + t + 1)}\right\} \ , \tag{7}$$

for every $t \geq 0$.

Remark 1.1. *The assumption* $\epsilon \leq V \leq M$ *yields the deterministic estimate*

$$C_\epsilon^{-1}\|x\| \leq \rho(x) \leq C_M\|x\| \ ,$$

which, in conjunction with (6) and (7), implies the inequality

$$\mathbb{P}\left\{|\rho(x) - \mu(x)| \geq t\right\} \leq C \exp\left\{-\frac{t^2}{C(\epsilon, M)\|x\|}\right\} \ .$$

Theorem 2. *Assume that the distribution of the potential is given by*

$$\mathbb{P}\left\{V(x) = a\right\} = \mathbb{P}\left\{V(x) = b\right\} = 1/2$$

for some $0 < a < b$*, and that* $d \geq 2$*. Then*

$$\operatorname{Var}\rho(x) \leq C_{a,b}\,\frac{\|x\|}{\log(\|x\| + 2)} \ . \tag{8}$$

2 Proof of Theorem 1

The proof of Theorem 1 is based on Talagrand's concentration inequality [13, 15]. We state this inequality as

Lemma 2.1 (Talagrand)**.** *Assume that* $\{V(x) \mid x \in \mathcal{X}\}$ *are independent random variables, the distribution of every one of which is supported in* $[0, M]$*. Then, for every convex (or concave) L-Lipschitz function* $f : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$*.*

$$\mathbb{P}\left\{f \geq \mathbb{E}f + t\right\} \leq C \exp\left\{-\frac{t^2}{CM^2L^2}\right\} \ ,$$

where $C > 0$ *is a constant.*

Denote $g(x) = G(0, x)$. To apply Lemma 2.1, we first compute the gradient of $\log g$, and then estimate its norm.

Lemma 2.2. *For any* $x, y \in \mathbb{Z}^d$*,*

$$\frac{\partial}{\partial V(y)} \log g(x) = -\frac{G(0, y)G(y, x)}{G(0, x)} \ .$$

Proof. Let $P_y = \delta_y \delta_y^*$ be the projector on the y-th coordinate. Set $H_h = H + hP_y$, $G_h = H_h^{-1}$. By the resolvent identity

$$G_h = G - hGP_yG_h \ ,$$

hence

$$\frac{d}{dh}\Big|_{h=0} G_h = -GP_yG$$

and

$$\frac{d}{dh}\Big|_{h=0} G_h(0,x) = -G(0,y)G(y,x) \ .$$

□

Our next goal is to prove

Proposition 2.3. *Suppose $V \geq \epsilon > 0$. Then*

$$\sum_y \left[\frac{G(0,y)G(y,x)}{G(0,x)} \right]^2 \leq A_\epsilon(\rho(x)+1) \ , \tag{9}$$

where A_ϵ depends only on ϵ.

The proof consists of two ingredients. The first one, equivalent to the triangle inequality for ρ, yields an upper bound on every term in (9).

Lemma 2.4. *For any $x, y \in \mathbb{Z}^d$,*

$$\frac{G(0,y)G(y,x)}{G(0,x)} \leq G(y,y) \leq C_\epsilon \ .$$

Proof. Let H_y be the operator obtained by erasing the edges that connect y to its neighbours, and let $G_y = H_y^{-1}$. By the resolvent identity,

$$G(0,x) = G_y(0,x) + \frac{1}{2d} \sum_{y' \sim y} G_y(0,y')G(y,x) \ .$$

In particular,

$$G(0,y) = \frac{1}{2d} \sum_{y' \sim y} G_y(0,y')G(y,y) \ .$$

Therefore

$$G(0,x) = G_y(0,x) + \frac{G(0,y)G(y,x)}{G(y,y)} \ .$$

□

The second ingredient is

Lemma 2.5. *For any* $x \in \mathbb{Z}^d$*,*

$$\sum_y \frac{G(0,y)G(y,x)}{G(0,x)} \leq C_\epsilon(\rho(x)+1) \ .$$

The proof of Lemma 2.5 requires two more lemmata. Denote

$$g_2(x) = G^2(0,x) = \sum_y G(0,y)G(y,x) \ , \quad u(x) = \frac{g_2(x)}{g(x)} \ .$$

Lemma 2.6. *For any* $x \in \mathbb{Z}^d$*,*

$$\sum_{y\sim x} \frac{g(y)}{2d(1+V(x))g(x)} = 1 - \frac{\delta(x)}{(1+V(0))g(0)} \tag{10}$$

and

$$u(x) = \sum_{y\sim x} u(y)\,\frac{g(y)}{2d(1+V(x))g(x)} + \frac{1}{1+V(x)} \ . \tag{11}$$

Proof. The first formula follows from the relation $Hg = \delta$, and the second one – from the relation $Hg_2 = g$. □

Set $\widetilde{\rho}(x) = \log \frac{G(0,0)}{G(0,x)}$.

Lemma 2.7. *For any* $x \in \mathbb{Z}^d$*,*

$$\widetilde{\rho}(x) \geq \sum_{y\sim x} \widetilde{\rho}(y)\,\frac{g(y)}{2d(1+V(x))g(x)} \\ + \log(1+V(x)) + \log\left(1 - \frac{1}{(1+V(0))g(0)}\right)\delta(x) \ .$$

Proof. For $x \neq 0$, (10) and the concavity of logarithm yield

$$\sum_{y\sim x} \frac{g(y)}{2d(1+V(x))g(x)} \log \frac{2d(1+V(x))g(x)}{g(y)} \leq \log(2d) \ .$$

Using (10) once again, we obtain

$$-\widetilde{\rho}(x) + \sum_{y\sim x} \widetilde{\rho}(y)\,\frac{g(y)}{2d(1+V(x))g(x)} + \log(1+V(x)) \leq 0 \ .$$

The argument is similar for $x = 0$. □

Proof of Lemma 2.5. Let $A \geq \log^{-1}(1+\epsilon)$. Then from Lemmata 2.6 and 2.7 the function $u_A = u - A\widetilde{\rho}$ satisfies

$$u_A(x) \leq \sum_{y \sim x} u_A(y) \, \frac{g(y)}{2d(1+V(x))g(x)} - A \log\left(1 - \frac{1}{(1+V(0))g(0)}\right) \delta(x) \ .$$

By a finite-volume approximation argument,

$$\max u_A(x) = u_A(0) \leq -\frac{A}{1 - \frac{1}{(1+V(0))g(0)}} \log\left(1 - \frac{1}{(1+V(0))g(0)}\right) \leq A'_\epsilon \ ,$$

whence

$$u(x) \leq A'_\epsilon + A\widetilde{\rho}(x) \leq C_\epsilon(1+\rho(x)) \ .$$

$\square$

Proof of Proposition 2.3. By Lemma 2.4 ,

$$\begin{aligned} L &= \sum_y \left[\frac{G(0,y)G(y,x)}{G(0,x)}\right]^2 \\ &\leq \max_y G(y,y) \sum_y \frac{G(0,y)G(y,x)}{G(0,x)} = \max_y G(y,y) \, u(x) \ . \end{aligned}$$

The inequality $V \geq \epsilon$ implies $G(y,y) \leq A''_\epsilon$, and Lemma 2.5 implies

$$u(x) \leq C_\epsilon(\rho(x)+1) \ .$$

$\square$

Next, we need

Lemma 2.8. *For any $x \in \mathbb{Z}^d$, $\log g(x)$, $\log \frac{G(0,x)}{G(0,0)}$, and $\log \frac{G(0,x)}{G(x,x)}$ are convex functions of the potential. Consequently,*

$$\rho(x) = -\frac{1}{2}\left[\log \frac{G(0,x)}{G(0,0)} + \log \frac{G(0,x)}{G(x,x)}\right]$$

is a concave function of the potential.

Proof. The first statement follows from the random walk expansion:

$$g(x) = \sum \frac{1}{1+V(x_0)} \frac{1}{2d} \frac{1}{1+V(x_1)} \frac{1}{2d} \cdots \frac{1}{2d} \frac{1}{1+V(x_k)} ,$$

where the sum is over all paths $w : x_0 = 0, x_1, \cdots, x_{k-1}, x_k = x$. Indeed, for every w

$$T_w = \log \frac{1}{1+V(x_0)} \frac{1}{2d} \frac{1}{1+V(x_1)} \frac{1}{2d} \cdots \frac{1}{2d} \frac{1}{1+V(x_k)}$$

is a convex function of V, hence also $\log g(x) = \log \sum_w e^{T_w}$ is convex.

To prove the second statement, observe that

$$G(0,x) = \frac{1}{2d} G(0,0) \sum_{y \sim 0} G_0(y,x) ,$$

where G_0 is obtained by deleting the edges adjacent to 0. Therefore

$$\log \frac{G(0,x)}{G(0,0)} = -\log(2d) + \log \sum_{y \sim 0} G_0(y,x) ;$$

for every y, $\log G_0(y,x)$ is a convex function of V, hence so is $\log \frac{G(0,x)}{G(0,0)}$. $\square$

Proof of Theorem 1. Denote $\rho_0(x) = \min(\rho(x), \mu(x))$. Then by Lemma 2.2 and Proposition 2.3

$$\|\nabla_V \rho_0(x)\|_2^2 \leq A_\epsilon(\mu(x)+1) ,$$

A_ϵ depends only on ϵ. By Lemma 2.8, ρ_0 is concave, therefore by Lemma 2.1

$$\mathbb{P}\left\{\rho(x) \leq \mu(x) - t\right\} \leq \exp\left\{ -\frac{t^2}{CM^2 A_\epsilon(\mu(x)+1)} \right\} .$$

Similarly, set $\rho_t(x) = \min(\rho(x), \mu(x)+t)$. Then

$$\|\nabla_V \rho_t(x)\|_2^2 \leq A_\epsilon(\mu(x)+t+1) ,$$

therefore

$$\begin{aligned} \mathbb{P}\left\{\rho(x) \geq \mu(x)+t\right\} &= \mathbb{P}\left\{\rho_t(x) \geq \mu(x)+t\right\} \\ &\leq \exp\left\{ -\frac{t^2}{CM^2 A_\epsilon(\mu(x)+t+1)} \right\} . \end{aligned}$$

$\square$

3 Proof of Theorem 2

The proof follows the strategy of Benjamini, Kalai, and Schramm [4]. Without loss of generality we may assume that $\|x\| \geq 2$; set $m = \lfloor \|x\|^{1/4} \rfloor + 1$.

Let

$$F = -\frac{1}{\#B} \sum_{z \in B} \log G(z, x+z) \ ,$$

where

$$B = B(0, m) = \{ z \in \mathbb{Z}^d \ | \ \|z\| \leq m \}$$

is the ball of radius m about the origin. According to Lemma 2.4,

$$G(0,x) \geq \frac{G(z, x+z) G(0,z) G(x, x+z)}{G(z,z) G(x+z, x+z)} \ ,$$

therefore $\rho(x) \leq F + C_{a,b} m$; similarly, $\rho(x) \geq F - C_{a,b} m$. It is therefore sufficient to show that

$$\operatorname{Var} F \leq C_{a,b} \frac{\|x\|}{\log \|x\|} \ .$$

We use another inequality due to Talagrand [14] (see Ledoux [11] for a semigroup derivation). Let $\mathcal{X}$ be a (finite or countable) set. Let $\sigma_x^+ : \{a,b\}^{\mathcal{X}} \to \{a,b\}^{\mathcal{X}}$ be the map setting the x-th coordinate to b, and $\sigma_x^- : \{a,b\}^{\mathcal{X}} \to \{a,b\}^{\mathcal{X}}$ –the map setting the x-th coordinate to a. Denote

$$\partial_x f = f \circ \sigma_x^+ - f \circ \sigma_x^- \ .$$

Lemma 3.1 (Talagrand)**.** *For any function f on $\{a,b\}^{\mathcal{X}}$,*

$$\operatorname{Var} f \leq C_{a,b} \sum_{x \in \mathcal{X}} \frac{\mathbb{E}|\partial_x f|^2}{1 + \log \frac{\mathbb{E}|\partial_x f|^2}{(\mathbb{E}|\partial_x f|)^2}} \ . \tag{12}$$

Let us estimate the right-hand side for $f = F$, $\mathcal{X} = \mathbb{Z}^d$. Denote

$$\sigma_x^t = t\sigma_x^+ + (1-t)\sigma_x^- \ ;$$

then

$$\partial_x F = \int_0^1 \frac{\partial F}{\partial V(x)} \circ \sigma_x^t \, dt \ .$$

According to Lemma 2.2,

$$\frac{\partial F}{\partial V(y)} = \frac{1}{\#B} \sum_{z \in B} \frac{G(z,y)G(y,x+z)}{G(z,x+z)} \ .$$

Therefore

$$\begin{aligned} \mathbb{E}\frac{\partial F}{\partial V(y)} \circ \sigma_y^t &= \mathbb{E}\frac{1}{\#B} \sum_{z \in B} \frac{G(z,y)G(y,x+z)}{G(z,x+z)} \circ \sigma_y^t \\ &= \mathbb{E}\frac{1}{\#B} \sum_{z \in B} \frac{G(0,y-z)G(y-z,x)}{G(0,x)} \circ \sigma_{y-z}^t \\ &= \mathbb{E}\frac{1}{\#B} \sum_{v \in y+B} \frac{G(0,v)G(v,x)}{G(0,x)} \circ \sigma_v^t \ . \end{aligned}$$

Lemma 3.2. *For any $Q \subset \mathbb{Z}^d$ and any $x', x \in \mathbb{Z}^d$,*

$$\sum_{v \in Q} \frac{G(x',v)G(v,x)}{G(x',x)} \leq C_a(\mathrm{diam}_\rho Q + 1) \leq C_{a,b}(\mathrm{diam}\, Q + 1) \ . \tag{13}$$

Let us first conclude the proof of Theorem 2 and then prove the lemma. Set $\delta = m^{-\frac{1}{2}}$, and let

$$A = \left\{ y \in \mathbb{Z}^d \,\middle|\, \mathbb{E}\left(\partial_y F\right)^2 \leq \delta\, \mathbb{E}\partial_y F \right\} \ .$$

Then the contribution of coordinates in A to the right-hand side of (12) is at most $C\delta\|x\|$ by Lemma 2.5. For y in the complement of A, Lemma 3.2 yields

$$\mathbb{E}\partial_y F \leq \frac{Cm}{\#B} \ ,$$

hence

$$\mathbb{E}\left(\partial_y F\right)^2 \geq \delta\, \mathbb{E}\partial_y F \geq \frac{\delta \# B}{Cm} \left(\mathbb{E}\partial_y F\right)^2 \ ,$$

and

$$\log \frac{\mathbb{E}\left(\partial_y F\right)^2}{\left(\mathbb{E}\partial_y F\right)^2} \geq \log \frac{\delta}{Cm} \geq \log(\|x\|/C')$$

by the inequality $\#B \geq Cm^2$ (which holds with d-independent C). The contribution of the complement of A to (12) is therefore at most $C' \frac{\|x\|}{\log\|x\|}$. Thus finally

$$\mathrm{Var} F \leq \frac{C''\|x\|}{\log\|x\|} \ .$$

$\square$

Proof of Lemma 3.2. For $Q \subset \mathbb{Z}^d$ and $x', x \in \mathbb{Z}^d$, set

$$u_Q(x', x) = \frac{(G \mathbb{1}_Q G)(x', x)}{G(x', x)} = \frac{\sum_{q \in Q} G(x', q) G(q, x)}{G(x', x)} .$$

Similarly to Lemma 2.6,

$$u_Q(x', x) = \sum_{y \sim x} u_Q(x', y) \frac{G(x', y)}{2d(1 + V(x)) G(x', x)} + \frac{\mathbb{1}_Q(x)}{1 + V(x)} .$$

By a finite-volume approximation argument, it is sufficient to prove the estimate (13) in a finite box. Then $\max_x u_Q(x', x)$ is attained for some $x_{\max} \in Q$. By symmetry, $\max_{x', x} u_Q(x', x)$ is attained when both x' and x are in Q. On the other hand, for $x', x \in Q$

$$u_Q(x', x) \leq u_{\mathbb{Z}^d}(x', x) \leq C(1 + \log \frac{1}{G(x', x)}) \leq C'(1 + \operatorname{diam}_\rho Q)$$

by Lemma 2.5. $\square$

Remark 3.3. *To extend Theorem 2 to the generality of the work of Benaïm and Rossignol [3], one may use the modified Poincaré inequality of [3] instead of Talagrand's inequality (12).*

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